

STABLE QUASIMAPS TO HOLOMORPHIC SYMPLECTIC QUOTIENTS

BUMSIG KIM

ABSTRACT. We apply the stable (twisted) quasimap construction to holomorphic symplectic quotients and obtain moduli spaces with symmetric obstruction theories.

1. INTRODUCTION

There are, so far, two classes of moduli examples which naturally carry symmetric obstruction theories:

- Moduli of stable objects in the abelian category of coherent sheaves on a Calabi-Yau threefold ([Th], see also [PT]).
- Moduli of stable objects in the abelian category of representations of a quiver with relations given by a superpotential ([Sz2]).

In this paper, we add one more such class:

- Moduli of stable objects in the abelian category of coherent \mathbf{M} -twisted quiver sheaves on a projective smooth curve C .

It arises as a curve counting on a holomorphic symplectic quotient described by a quiver. Note that the natural perfect obstruction theory on the mapping space $\mathrm{Mor}(C, Y)$ is symmetric when the target Y is holomorphic symplectic and the domain curve C is an elliptic curve, since for a map $f : C \rightarrow Y$,

$$\begin{aligned} \mathrm{ob}(f)^\vee &:= H^1(C, f^*\mathcal{T}_X)^\vee \cong H^0(C, f^*\Omega_X \otimes \omega_C) \\ &\cong H^0(C, f^*\mathcal{T}_X) = \mathrm{def}(f), \end{aligned}$$

where $\mathrm{ob}(f)$ is the obstruction space at f and $\mathrm{def}(f)$ is the 1st order deformation space at f . When Y is given by a holomorphic symplectic quotient of an affine variety X by a complex reductive Lie group G action, we can apply the quasimap construction of [CK, CKM] in order to ‘compactify’ $\mathrm{Mor}(C, Y)$. The advantage of the construction compared to the stable map construction is that we can keep the fixed

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domain curve so that the natural extension of the obstruction theory still remains symmetric (Proposition 3.6). When X carries a torus action commuting with G such that the fixed loci $X//G^S$ is proper over \mathbb{C} , then the induced fixed loci on the moduli space of stable quasimaps is also proper over \mathbb{C} . Therefore, one can obtain well-defined localization residue ‘invariants’ in the case.

This method can be applied to the case when Y is a holomorphic symplectic quotient of a double quiver. In this case, a quasimap is nothing but a quiver bundle on C . Using the idea of twisted quiver bundles, we can obtain the corresponding notion of twisted quasimap. We show that the moduli space of stable \mathbf{M} -twisted quasimaps carries a symmetric obstruction theory (Theorem 5.9) and the stability as quasimaps coincides with the asymptotic stability of the slope stability of quiver bundles (Proposition 5.11). In the case of ADHM quiver, these facts have shown by Diaconescu in [Di], which is, together with the quasimap construction [CK, CKM], the main source of inspiration for this work. We also show that with respect to a slope stability, the moduli space of stable twisted quiver bundles carries a natural symmetric obstruction theory (Theorem 5.9).

The typical examples for double quivers in our framework can be ADHM quiver ([Di, CDP1, CDP2]) and the framed ADE quivers. We will study the wall-crossing phenomena elsewhere, showing that the conditions corresponding to (a) (the moduli stack is analytic-locally the critical locus of a holomorphic function on a smooth complex domain) and (b) (the Euler form is numerical) in §1.5 [JS] holds in our setting.

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2. HOLOMORPHIC SYMPLECTIC QUOTIENTS

2.1. Symplectic quotients. We set up holomorphic symplectic quotients for our purpose. Let X be a smooth affine variety over \mathbb{C} equipped with a holomorphic symplectic form

$$\omega : \mathcal{T}_X \otimes \mathcal{T}_X \rightarrow \mathcal{O}_X.$$

Suppose that a complex reductive, connected Lie group G acts on X as a hamiltonian action, i.e., the action preserves ω and there is a so-called *complex* moment map $\mu : X \rightarrow \mathcal{G}^*$. The moment map is, by definition,

a G -equivariant algebraic morphism such that for every tangent vector ξ of X and every $g \in \mathcal{G}$

$$(2.1) \quad \langle d\mu(\xi), g \rangle = \omega(d\alpha(g), \xi)$$

where $d\alpha$ is the derivative of the action map $\alpha : G \rightarrow \text{Aut}(X)$. Here G acts on \mathcal{G}^* by the coadjoint representation Ad^* .

Choose an Ad^* -invariant element λ in \mathcal{G}^* . For a character $\chi \in \text{Hom}(G, \mathbb{C}^\times)$, let \mathbb{C}_χ be the one-dimensional representation \mathbb{C} associated to χ and let L be the linearization $\mu^{-1}(\lambda) \times \mathbb{C}_\chi$. The *holomorphic symplectic quotient* $X //_{\lambda, L} G$ is defined to be a GIT quotient

$$\mu^{-1}(\lambda) //_{L} G := \mathbf{Proj} \left(\bigoplus_{l \geq 0} H^0(\mu^{-1}(\lambda), L^l)^G \right).$$

We will use the following conventions (see [Ne]).

Definition 2.1. A point p in $\mu^{-1}(\lambda)$ is called *semistable* if there is $s \in H^0(X, L^l)^G$ for some $l > 0$ such that $s(p) \neq 0$. The semistable point p is called *stable* if the stabilizer G_p is finite and the action of G on $\{q : s(q) \neq 0\}$ is closed (i.e., every orbit is closed in $\{q : s(q) \neq 0\}$).

For a 1-parameter subgroup, i.e., a homomorphism $\lambda : \mathbb{C}^\times \rightarrow G$, denote the exponent m of $t^m = \chi(\lambda(t))$ by $\langle \chi, \lambda \rangle$. Then, there is a numerical criterion for semistable and stable points adapting the Hilbert-Mumford criterion.

Proposition 2.2. ([Ki]) *Suppose that the G -action on X is linear in the sense that there is a G -equivariant closed embedding of X into a G -linear space. A point p is semistable (resp. stable) if $\langle \chi, \lambda \rangle \geq 0$ (resp. $\langle \chi, \lambda \rangle > 0$) for any nontrivial 1-parameter subgroup λ for which the limit of $\lambda(t) \cdot p$ as $t \rightarrow 0$ exists.*

In the above, the ‘sign’ convention is correct since $(g \cdot s)(p) = \chi(g)^l s(g^{-1} \cdot p)$ if $s \in H^0(\mu^{-1}(\lambda), L^l)$ so that

$$\bigoplus_{l \geq 0} H^0(\mu^{-1}(\lambda), L^l)^G = \mathbb{C}[\mu^{-1}(\lambda) \times \mathbb{C}_{\chi^{-1}}]^G,$$

where the right hand side denotes the G -invariant part of the affine coordinate ring of $\mu^{-1}(\lambda) \times \mathbb{C}$. Later we will assume that there are no strictly semistable points so that GIT quotient $\mu^{-1}(\lambda) //_{L} G$ is an orbit space $\mu^{-1}(\lambda)^s / G$.

The equation (2.1) shows that the isotropy group G_p of a point $p \in X$ is a finite group if and only if $d\mu|_{T_p X} : T_p X \rightarrow \mathcal{G}^*$ is surjective if and only if p is a regular point of the moment map μ . Hence, the stack quotient

$[\mu^{-1}(\lambda)^s/G]$ of the stable locus $\mu^{-1}(\lambda)^s$ is a holomorphic symplectic stack.

2.2. Symmetry. The derivatives of α and μ together give rise to a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G} \otimes \mathcal{O}_X & \xrightarrow{d\alpha} & \mathcal{T}_X & \xrightarrow{d\mu} & \mathcal{G}^* \otimes \mathcal{O}_X & \longrightarrow & 0 \\
\downarrow & & \downarrow \text{id} & & \downarrow \omega & & \downarrow -\text{id} & & \downarrow \\
0 & \longrightarrow & \mathcal{G} \otimes \mathcal{O}_X & \xrightarrow{d\mu^\vee} & \Omega_X & \xrightarrow{d\alpha^\vee} & \mathcal{G}^* \otimes \mathcal{O}_X & \longrightarrow & 0
\end{array}$$

of G -sheaves due to equation (2.1). Note that the first horizontal of the diagram is a G -equivariant complex after the restriction to $\mu^{-1}(\lambda)$. Here, the G -action on \mathcal{G} is the adjoint representation Ad . The complex will be considered as a 3-term perfect complex

$$F := \left[\mathcal{G} \otimes \mathcal{O}_X \xrightarrow{d\alpha} \mathcal{T}_X \xrightarrow{d\mu} \mathcal{G}^* \otimes \mathcal{O}_X \right]_{|_{\mu^{-1}(\lambda)}}$$

supported at $[-1, 1]$.

Note that the commutative diagram shows that

$$(2.2) \quad F^\vee \cong F$$

as complexes of G -sheaves. Note also that $F|_{\mu^{-1}(\lambda)^s}$ descends to the tangent sheaf of the stack quotient $[\mu^{-1}(\lambda)^s/G]$ upto quasi-isomorphisms (that is, the monad $F|_{\mu^{-1}(\lambda)^s}$ is quasi-isomorphic to the pullback of $\mathcal{T}_{[\mu^{-1}(\lambda)^s/G]}$). Hence, F is a locally free extension of $F|_{\mu^{-1}(\lambda)^s}$ which plays the rôle of $\mathcal{T}_{[\mu^{-1}(\lambda)^s/G]}$ on $\mu^{-1}(\lambda)^s$.

3. STABLE QUASIMAPS

3.1. Quasimaps. The notion of stable quasimaps appeared for a compactification of maps from C to a GIT quotient, in this paper which is $\mu^{-1}(\lambda)//G$. If there are no strictly semistable points and all isotropy groups are trivial, then such a map exactly amounts to a principal G -bundle P on C with a G -equivariant map $u : P \rightarrow \mu^{-1}(\lambda)^s$. We recall the precise definition of stable quasimaps with a fixed domain curve C .

Definition 3.1. By a *principal G -bundle* $\pi : P \rightarrow Y$ on a scheme Y , we mean a scheme P with a free *left* G -action which is étale locally trivial, i.e, there is an étale surjective morphism from a scheme Y' to Y making the pullback $P \times_Y Y'$ isomorphic to $G \times Y'$ as a G -space over Y' . By a *morphism between two principal bundles* on Y , we mean a G -equivariant morphism over Y .

If P is a principal bundle with right action as usual, then we will consider it as a principal bundle with left action through $G \rightarrow G$, $g \mapsto g^{-1}$.

Definition 3.2. ([CK, CKM]) A pair (P, u) is called a *quasimap* to $\mu^{-1}(\lambda)//G$ with a *rigid* domain curve C if P is a principal G -bundle on a projective smooth curve C and u is a section of a fiber bundle $P \times_G \mu^{-1}(\lambda)$. A quasimap (P, u) to $\mu^{-1}(\lambda)//_L G$ is called *stable* with respect to L if the preimage $u^{-1}(P \times_G (\mu^{-1}(\lambda))^u)$ of the L -unstable locus $\mu^{-1}(\lambda)^u$ is finite points.

Often, we will consider u also as a G -equivariant map from P to $\mu^{-1}(\lambda)$. Note that the definition of quasimaps does not require a choice of linearizations. Two quasimaps (P, u) and (P', u') are considered isomorphic if there is an isomorphism

$$\begin{array}{ccc} P & \xrightarrow{\phi} & P' \\ & \searrow & \swarrow \\ & C & \end{array}$$

of principal G -bundles preserving sections, that is, $\tilde{\phi} \circ u = u'$ where $\tilde{\phi} : P \times_G \mu^{-1}(\lambda) \rightarrow P' \times_G \mu^{-1}(\lambda)$ is the map induced from ϕ . Define a degree class β of (P, u) as the homomorphism

$$\beta : \text{Pic}^G(\mu^{-1}(\lambda)) \rightarrow \mathbb{Z}, \quad L' \mapsto \deg u^*(P \times_G L').$$

Later we will replace $\text{Pic}^G(\mu^{-1}(\lambda))$ by $\text{Pic}^G(X)$. The moduli stack $Qmap := Qmap(\mu^{-1}(\lambda)//_L G, \beta, C)$ of stable quasimaps with a rigid domain C and a fixed degree β is constructed in [CKM] as a DM stack, proper over the affine quotient $\mathbf{Spec} \mathbb{C}[\mu^{-1}(\lambda)]^G$. Here the map from $Qmap$ to $\mathbf{Spec} \mathbb{C}[\mu^{-1}(\lambda)]^G$ is obtained by assignment:

$$(P, u) \mapsto \text{Im}(C \xrightarrow{[u]} [\mu^{-1}(\lambda)/G] \rightarrow \mathbf{Spec} \mathbb{C}[\mu^{-1}(\lambda)]^G),$$

where the composite is a constant map since C is projective and the affine quotient is affine. The stack $Qmap$ is in fact an algebraic space since there are no nontrivial automorphisms so that the moduli DM stack coincides with its coarse moduli space.

3.2. Symmetric obstruction theory. Since X is smooth, the tangent complex $\mathbb{L}_{\mu^{-1}(\lambda)}^\vee$ of the complete intersection $\mu^{-1}(\lambda)$ is quasi-isomorphic to

$$[\mathcal{T}_X|_{\mu^{-1}(\lambda)} \rightarrow \mathcal{G}^* \otimes \mathcal{O}_{\mu^{-1}(\lambda)}].$$

Its descendant on the universal family $\mathcal{P} \times_G \mu^{-1}(\lambda)$ is the relative tangent complex \mathbb{L}_ρ^\vee , where \mathcal{P} is the universal principal G -bundle on the

universal curve \mathcal{C} of $Qmap$ and ρ is the projection from $\mathcal{P} \times_G \mu^{-1}(\lambda)$ to \mathcal{C} .

The deformation space and an obstruction space of sections u with fixed P are $H^i(C, u^*(\mathbb{L}_\rho^\vee)|_{C \times (P, u)})$, $i = 0, 1$, respectively. They are shown to formulate a relative perfect obstruction theory in [CKM], provided with

- Condition 3.3.* (1) There is no strictly semistable locus of $\mu^{-1}(\lambda)$.
 (2) The stable locus $\mu^{-1}(\lambda)^s$ is smooth.
 (3) The G -action on $\mu^{-1}(\lambda)^s$ is free.

The above condition will be assumed from now on unless stated otherwise. Denote the natural maps by

$$(3.1) \quad \mathcal{P} \times_G \mu^{-1}(\lambda) \begin{array}{c} \xrightarrow{\rho} \\ \xleftarrow{u} \end{array} \mathcal{C} = C \times Qmap \xrightarrow{\pi} Qmap \xrightarrow{\phi} \mathfrak{Bun}_G(C),$$

where $\mathfrak{Bun}_G(C)$ is the smooth algebraic stack of principal G -bundles on C .

Theorem 3.4. ([CKM]) *There is a natural relative perfect obstruction theory*

$$(R^\bullet \pi_* u^* \mathbb{L}_\rho^\vee)^\vee \rightarrow \mathbb{L}_{Qmap/\mathfrak{Bun}_G(C)}$$

for $Qmap$, where $\mathbb{L}_{Qmap/\mathfrak{Bun}_G(C)}$ is the two-term truncation of the relative cotangent complex.

Let \mathcal{F} denote the complex on $\mathcal{P} \times_G \mu^{-1}(\lambda)$ associated to the perfect complex F . In what follows, we slightly abuse notation by identifying locally free sheaves and their associated vector bundles. Since the cotangent complex $\phi^* \mathbb{L}_{\mathfrak{Bun}_G(C)}[1]$ is

$$(3.2) \quad (R^\bullet \pi_* \text{Ad} \mathcal{P})^\vee \cong (R^\bullet \pi_* u^*(\mathcal{P} \times_G (\mu^{-1}(\lambda) \times \mathcal{G})))^\vee,$$

the induced homomorphism $(R^\bullet \pi_* u^* \mathcal{F})^\vee \rightarrow \mathbb{L}_{Qmap}$ becomes a (absolute) perfect obstruction theory, which will be shown to be symmetric in the below. Here the isomorphism (3.2) is inherited from the fiber square

$$\begin{array}{ccc} \mathcal{P} \times \mu^{-1}(\lambda) & \longleftarrow & \mathcal{P} \times \mu^{-1}(\lambda) \times \mathcal{G} \\ \uparrow (\text{id}, u) & & \uparrow (\text{id}, u) \times \text{id} \\ \mathcal{P} & \longleftarrow & \mathcal{P} \times \mathcal{G}. \end{array}$$

and the induced homomorphism is obtained by one of the axioms of triangulated categories.

Definition 3.5. ([BF]) A perfect obstruction theory $E \rightarrow \mathbb{L}_M$ for a finite type DM-stack M is called *symmetric* if it is endowed with an isomorphism $\alpha : E \rightarrow E^\vee[1]$ in the derived category of coherent sheaves on M such that $\alpha^\vee[1] = \alpha$ holds, where \mathbb{L}_M is the two-term truncation of the cotangent complex of M .

Proposition 3.6. *Assume Conditions 3.3. If C is an elliptic curve, then the perfect obstruction theory*

$$(R^\bullet \pi_* u^* \mathcal{F})^\vee \rightarrow \mathbb{L}_{Qmap}$$

for $Qmap(X//G, \beta, C)$ is a symmetric obstruction theory.

Proof. By Grothendieck duality and (2.2), $(R^\bullet \pi_* u^* \mathcal{F})^\vee \cong R^\bullet \pi_* ((u^* \mathcal{F}^\vee) \otimes \omega_C[1]) \cong R^\bullet \pi_* u^* \mathcal{F}[1]$. Denote by α the composite of the isomorphisms. We need to check that $\alpha^\vee = \alpha[-1]$. This can be seen by $\text{tr}(a, ib) = \text{tr}(i^\vee a, b)$ where tr is a trace map and i is the isomorphism $\mathcal{F} \rightarrow \mathcal{F}^\vee$. \square

Remark 3.7. Let $E = (R^\bullet \pi_* u^* \mathcal{F})^\vee$. Then in the proof of Proposition 3.6 we show that there is an isomorphism $E \rightarrow E^\vee[1]$. This in turn together with the fact that there are no nontrivial infinitesimal automorphisms, implies that E is a two-term perfect complex in derived category of coherent sheaves. This, in particular, shows that (2) and (3) in Conditions 3.3 are unnecessary. The Condition (1) is needed to establish that there are no nontrivial infinitesimal automorphisms.

4. QUASIMAPS TO SYMPLECTIC QUIVER VARIETIES

4.1. Nakajima type quiver varieties. Let Q be a finite quiver, which means that it is equipped with two finite sets Q_0, Q_1 and two maps $t, h : Q_1 \rightarrow Q_0$. We call Q_0 (resp. Q_1) the vertex (resp. arrow) set of Q and ta (resp. ha) the tail (resp. head) of arrow $a \in Q_1$. Let \overline{Q} be the double quiver of Q . It is defined as follows. The vertex set \overline{Q}_0 of \overline{Q} is exactly Q_0 . For each arrow a in Q_1 , create exactly two associated arrows a^+, a^- of \overline{Q} , by making the head (resp. tail) of a^+ (resp. a^-) = the head (resp. tail) of a . Fix a subset Q'_0 of Q_0 and let $Q''_0 := Q_0 \setminus Q'_0$. Given a dimension vector $v = (v_i) \in \mathbb{N}^{Q_0}$, let

$$\text{Rep}(\overline{Q}, v) := \bigoplus_{a \in Q_1} (\text{Hom}(\mathbb{C}^{v_{ta^+}}, \mathbb{C}^{v_{ha^+}}) \oplus \text{Hom}(\mathbb{C}^{v_{ta^-}}, \mathbb{C}^{v_{ha^-}})).$$

After $\text{Rep}(\overline{Q}, v)$ being canonically identified with the total space of the cotangent bundle of

$$\bigoplus_{a \in Q_1} \text{Hom}(\mathbb{C}^{v_{ta^+}}, \mathbb{C}^{v_{ha^+}}),$$

$X := \text{Rep}(\overline{Q}, v)$ can be regarded as a holomorphic symplectic manifold with the canonical holomorphic symplectic form ω . The linear symplectic form is defined by

$$\omega(A, B) = \sum_a \text{tr} A_{a+} B_{a-} - \text{tr} A_{a-} B_{a+}$$

for tangent vectors $A, B \in T_x \text{Rep}(\overline{Q}, v) = \text{Rep}(\overline{Q}, v)$. Let

$$G := \prod_{i \in Q'_0} GL_{v_i}(\mathbb{C}).$$

Then, there is a natural hamiltonian G -action on X with a moment map

$$\mu(x) = \sum_{a \in \overline{Q}_1 : t\overline{a} \in Q'_0} (-1)^{|a|} x_a x_{\overline{a}} \quad \text{for } x = (x_a)_{a \in \overline{Q}_1} \in \text{Rep}(\overline{Q}, v)$$

where \overline{a} denotes the opposite arrow of a and $|b^+| = 0$, $|b^-| = 1$ for $b \in Q_1$. Here we identified \mathcal{G} with its dual by trace. The equation (2.1) can be checked easily since for $g \in \mathcal{G}$, $d\alpha(g)$ is the linear vector field $(g_{ha}x_a - x_ag_{ta})_a \in \text{Rep}(\overline{Q}, v)$.

Let

$$\lambda = \sum_{i \in Q'_0} \lambda_i \text{Id}_{\text{End}_{v_i}(\mathbb{C})} \in \bigoplus_{i \in Q'_0} \text{End}_{v_i}(\mathbb{C}), \quad \lambda_i \in \mathbb{C}$$

and choose a character $\chi = (\theta_i) \in \mathbb{Z}^{Q'_0}$ of G by sending $g \in G$ to $\prod_{i \in Q'_0} (\det g_i)^{\theta_i} \in \mathbb{C}^\times$. This defines a linearization $L = \mu^{-1}(\lambda) \times_{\chi} \mathbb{C}_\chi$. We consider the holomorphic symplectic quotient $\mu^{-1}(\lambda) //_{\chi} G$. Since there is no difference between GIT quotients with respect to χ and $m\chi$ for any positive integer m , we allow that θ_i are rational numbers.

On X , there is an action by

$$\mathbf{T} = (\mathbb{C}^\times)^{\overline{Q}_1}$$

which commutes with G -action. It is defined by

$$t \cdot \phi = (t_a \phi_a)_{a \in \overline{Q}_1} \quad \text{for } \phi \in X, \quad t \in \mathbf{T}.$$

4.2. Twisted quasimap. For each arrow $a \in Q_1$, fix two line bundles M_{a^\pm} on C with a fixed isomorphism $M_{a^+} \otimes M_{a^-} \rightarrow \omega_C$. In the below, we identify $M_{a^+} \otimes M_{a^-} \cong \omega_C$ and for $M_{a^-} \otimes M_{a^+} \cong \omega_C$ we will use the natural isomorphism followed by the given one: $M_{a^-} \otimes M_{a^+} \rightarrow M_{a^+} \otimes M_{a^-} \rightarrow \omega_C$.

Definition 4.1. A pair (P, u) is called a **M**-twisted quasimap to $X //_{\lambda=0} G$ if:

- P is a principal G -bundle on C .

- $u = (u_a)_{a \in \overline{Q}_1}$ where

$$u_a \in \Gamma(C, P \times_G \operatorname{Hom}(\mathbb{C}^{v_{ta}}, \mathbb{C}^{v_{ha}}) \otimes M_a).$$

- The section u satisfies the ‘moment’ map equation (for $\lambda = 0$):

$$\sum_{i \in Q'_0} \sum_{a \in \overline{Q}_1 : t\overline{a} = i} (-1)^{|a|} (u_a \otimes \operatorname{Id}_{M_{\overline{a}}}) \otimes u_{\overline{a}} = 0.$$

A \mathbf{M} -twisted quasimap (P, u) to $X \mathbin{\mathbb{H}}_{\lambda=0} G$ is called a *stable \mathbf{M} -twisted quasimap* to $X \mathbin{\mathbb{H}}_{\lambda=0} G$ with a rigid domain curve C if u hits unstable locus only at finite points of C .

In the definition of stable \mathbf{M} -twisted quasimaps, we need explanations. After fixing $z \in C$ and identifications of fibers: $(M_a)|_z = \mathbb{C}$ for $a \in Q_1$ and $(\omega_C)|_z = \mathbb{C}$, it makes sense whether u hits unstable locus at z . Since the different choices of identification make points $u(z) \in X$ in the same \mathbf{T} -orbit and \mathbf{T} action commutes with G action, the (semi)stability of $u(z)$ is independent of the choices. Therefore the statement that u hits unstable locus only at finite points of C is well-defined. In the above definition, we let $\lambda = 0$ to construct $X \mathbin{\mathbb{H}} G$.

We define the degree of (P, u) as a homomorphism β from the character group $\operatorname{Hom}(G, \mathbb{C}^\times)$ to \mathbb{Z} by $\beta(\nu) = \deg(P \times_G \mathbb{C}_\nu)$. Similarly to the untwisted case, we conclude this.

Proposition 4.2. *Assume (1) and (3) Condition 3.3. The moduli stack*

$$Qmap(X \mathbin{\mathbb{H}}_{\lambda=0, \chi} G, \mathbf{M}, \beta, C)$$

of stable \mathbf{M} -twisted quasimaps with degree β to $X \mathbin{\mathbb{H}} G$ is a proper DM stack over $\mathbf{Spec} \mathbb{C}[\mu^{-1}(0)]^G$ when $\chi = (1, \dots, 1)$.

Proof. In the construction of the moduli stack as a DM stack of finite type, only nontrivial part is the boundedness, which will be given in the proof of Theorem 5.9.

We prove the valuative criterion for properness. Let S° be a punctured smooth curve of a smooth curve S and let (P, u) be a stable twisted quasimap on $C \times S^\circ$. Suppose that the induced map from S° to $\mathbf{Spec} \mathbb{C}[\mu^{-1}(0)]^G$ is extendable to a map from S .

Existence: Denote by B the locus of $C \times S^\circ$ where u hits unstable locus and let $\{U_i\}$ be a finite open covering of C with trivializations of ω_C and $M_a \forall a$ on each U_i . After fixing trivializations, consider a regular map from $U_i \times S^\circ \setminus B$ to $X \mathbin{\mathbb{H}} G$. Since $X \mathbin{\mathbb{H}} G \rightarrow \mathbf{Spec} \mathbb{C}[\mu^{-1}(0)]^G$ is proper, the map can be extendable to $[u_i] : U_i \times S \setminus (B \amalg Z) \rightarrow X \mathbin{\mathbb{H}} G$, where Z consists of finite points. Therefore there are corresponding extensions (P_i, u_i) of (P, u) on $U_i \times S \setminus (B \amalg Z)$. We claim that P_i can

be extendable to a principal bundle \tilde{P}_i on $U_i \times S$. This can be seen by considering a regular map extension near Z of $[u_i]$ by blowing up along Z and then push-forwarding the associated vector bundle under the blowdown map to obtain coherent sheaves on $U_i \times S$. Take the double dual E_i of the sheaves which are locally free on $U_i \times S$ and let \tilde{P}_i be the associated principal bundle of E_i . The natural isomorphism between \tilde{P}_i and \tilde{P}_j on $U_i \cap U_j \times S \setminus Z$ is extendable on $C \times S$ since Z is isolated points. Denote by \overline{P} the glued principal bundle on $C \times S$. We also have a glued section of u_i on $C \times S \setminus Z$ and have the extended section \overline{u} of $\overline{P} \times_G V$ on $C \times S$ by Hartogs' theorem. By construction, the extension $(\overline{P}, \overline{u})$ is clearly a stable twisted quasimap to $X//G$ over S .

Uniqueness: If there are two stable extensions $(\overline{P}, \overline{u})$ and $(\overline{P}', \overline{u}')$, then they must be isomorphic on $C \times S \setminus Z$ for some finite set Z due to the induced rational maps from $U_i \times S$ to $X//G$. The isomorphism is in turn extendable on $C \times S$. \square

4.3. Obstruction theory. Let $\mathcal{E} \otimes \mathbf{M}$ denote the complex

$$(4.1) \quad 0 \rightarrow \mathrm{Ad}\mathcal{P} \xrightarrow{d\alpha} \bigoplus_{a \in Q_1} (\mathcal{H}om(\mathcal{V}_{ta^+}, \mathcal{V}_{ha^+}) \otimes \pi_C^* M_{a^+} \\ \oplus \mathcal{H}om(\mathcal{V}_{ta^-}, \mathcal{V}_{ha^-}) \otimes \pi_C^* M_{a^-}) \\ \xrightarrow{d\mu} \mathrm{Ad}\mathcal{P}^\vee \otimes \pi_C^* \omega_C \rightarrow 0$$

on \mathcal{C} , where $\mathcal{V}_i = \mathcal{P} \times_G \mathbb{C}^{v_i}$ and π_C is the projection $C \times Qmap \rightarrow C$ (for the notation of others see (3.1)). Here $\mathrm{Ad}\mathcal{P} = \mathcal{P} \times_G \mathcal{G} = (\mathcal{P} \times_G \mathcal{G}^*)^\vee$.

Proposition 4.3. *Assume (1) in Condition 3.3. The stack*

$$Qmap(X//G, \mathbf{M}, \beta, C)$$

comes equipped with a symmetric obstruction theory

$$(R^\bullet \pi_* \mathcal{E} \otimes \mathbf{M})^\vee \rightarrow \mathbb{L}_{Qmap}.$$

Proof. It is straightforward to check that the argument in subsection 3.2 proving Proposition 3.6 with ‘twist’ by \mathbf{M} works fine. By Remark 3.7, (2) and (3) of Condition 3.3 are not necessary. \square

4.4. S-action. Typically, the quotient $X//G$ is not proper over \mathbb{C} . However, the fixed locus $X//G^{\mathbf{S}}$ might be proper, where $\mathbf{S} = (\mathbb{C}^\times)^{Q_1} \subset_i \mathbf{T}$ by $i(t)_{a^+} = t_a$ and $i(t)_{a^-} = t_a^{-1}$ for $a \in Q_1$. From \mathbf{T} action on X , \mathbf{S} action on X is induced. Note that the moment map is invariant under \mathbf{S} action.

The \mathbf{S} fixed loci $(\mu^{-1}(0)//_X G)^{\mathbf{S}}$ can be considered as the induced GIT quotient

$$\overline{p^{-1}((\mu^{-1}(0)//_X G)^{\mathbf{S}})}//_X G$$

where $p : \mu^{-1}(0)^s \rightarrow \mu^{-1}(0) \parallel_{\chi} G$ is the projection. Then note that

$$Qmap(\mu^{-1}(0) \parallel_{\chi} G, \mathbf{M}, \beta, C)^{\mathbf{S}} = Qmap((\mu^{-1}(0) \parallel_{\chi} G)^{\mathbf{S}}, \mathbf{M}, \beta, C)^{\mathbf{S}},$$

under the induced actions on the moduli spaces. Note that (4.1) is \mathbf{S} -equivariant. Hence by Proposition 4.3 and the virtual localization of Graber and Pandharipande ([GP]), we conclude this.

Corollary 4.4. *Assume (1) and (3) of Condition 3.3. There is a natural symmetric obstruction theory for $Qmap^{\mathbf{S}}$. If the fixed loci $(\mu^{-1}(0) \parallel_{\chi} G)^{\mathbf{S}}$ is proper over \mathbb{C} , so is $Qmap^{\mathbf{S}}$.*

Proof. The induced obstruction theory is obtained from the dual of the Čech resolution \mathbf{C} of (4.1) equipped with the natural \mathbf{S} -action which is nontrivial only on the middle term. Note that \mathbf{S} -equivariantly $\mathbf{C} \cong \mathbf{C}^{\vee}[-1]$ by Serre duality. Hence, the perfect obstruction theory for $Qmap^{\mathbf{S}}$ induced from $\mathbf{C}^{\mathbf{S}}$ satisfies the desired symmetry since $\mathbf{C}^{\mathbf{S}} \cong (\mathbf{C}^{\mathbf{S}})^{\vee}[-1]$. More precisely, let $\text{Spec} A$ be affine open ‘subset’ of $Qmap$ in étale topology. Choose a finite affine open covering \mathfrak{U} of $C \times \text{Spec} A$ and let \mathbf{C} be the total Čech complex of (4.1) with respect to \mathfrak{U} . By choosing a suitable affine open covering, we may assume that the only nontrivial terms are \mathbf{C}^i , $i = -1, 0, 1, 2$. Let $(\mathbf{C}^{\mathbf{S}}, \partial^i)$ be the \mathbf{S} fixed part of \mathbf{C} . Choose a finitely generated free A -module F^1 with a A -module surjection from F^1 to $\text{Ker}(\partial^1)$ and define $F^0 = \{(a, b) \in \text{Coker}(\partial^{-1}) \oplus F^1 \mid a = b \text{ in } \text{Ker}(\partial^1)\}$. Then it is easy to check that the induced complex F^{\bullet} is quasi-isomorphic to $\mathbf{C}^{\mathbf{S}}$. Consider F as the complex of associated \mathcal{O}_A -sheaves on $\text{Spec} A$. Note that F^0 is locally free since F^1 is locally free and the difference of dimension of fibers of F^i is constant (zero in this case). This F is the induced perfect obstruction theory for $Qmap^{\mathbf{S}}$ with the symmetry property $F \cong F^{\vee}[-1]$. \square

5. STABILITIES ON QUIVER BUNDLES

5.1. King’s stability. The stability with respect to linearization χ we used in the previous section can be reinterpreted as a Bridgeland’s stability condition on a suitable abelian category of representations of the path algebra $\mathbb{C}\overline{Q}$ with relations.

The path algebra is a \mathbb{C} -algebra spanned by, as a \mathbb{C} vector space, all finite paths $a_n \dots a_1$ of consecutive arrows and an extra arrow e_i , for each $i \in Q_0$, where $a_l \in \overline{Q}_1$ and $ha_l = ta_{l+1}$ for all l . The product is given by a sort of compositions. Namely, $(a_{j_m} \dots a_{j_1}) \cdot (a_{k_n} \dots a_{k_1})$ is $a_{j_n} \dots a_{j_1} a_{k_n} \dots a_{k_1}$ if $ha_{k_n} = ta_{j_1}$, 0 otherwise. The generators are subject with relations: $e_i^2 = e_i$; $e_i a$ is a if $ha = i$, 0 otherwise; and ae_i is a if $ta = i$, 0 otherwise. Impose one more relation coming from the moment

map together with $\lambda \in \mathbb{C}^{Q'_0}$:

$$\sum_{a \in \overline{Q}_1: t\bar{a} \in Q'_1} (-1)^{|a|} a\bar{a} = \sum_{i \in Q'_0} \lambda_i e_i$$

which will be denoted symbolically by $\mu - \lambda = 0$.

Denote by $(\mu - \lambda)$ the two-sided ideal generated by $\mu - \lambda$. Note that a $\mathbb{C}\overline{Q}/(\mu - \lambda)$ -module V amounts to data $(V_i, \phi_a)_{i \in Q_0, a \in \overline{Q}_1}$ where V_i is a \mathbb{C} -vector space and ϕ_a is a homomorphism from V_{ta} to V_{ha} subject to the condition coming from $\mu - \lambda = 0$. A homomorphism from a module (V_i, ϕ_a) to another (V'_i, ϕ'_a) is nothing but a collection $(\varphi_i)_{i \in Q_0}$ of linear maps $\varphi_i : V_i \rightarrow V'_i$ making $\phi'_a \circ \varphi_{ta} = \varphi_{ha} \circ \phi_a$ for every $a \in \overline{Q}_1$.

Now we are ready to reformulate the χ -stability of $V \in X$ following [Ki]. Suppose that $v_0 \neq 0$. Let $\theta_0 = -(\sum_{i \in Q'_0} \theta_i v_i)/v_0$ and for a $\mathbb{C}\overline{Q}/(\mu - \lambda)$ -module W let

$$\theta(W) = \theta_0 \dim W_0 + \sum_{i \in Q'_0} \theta_i \dim W_i,$$

where $W_0 := \oplus_{i \in Q'_0} \dim W_i$. Note that $\theta(V) = 0$.

Theorem 5.1. *Suppose that $\dim V_i \neq 0$ for every $i \in Q'_0$. The followings are equivalent.*

- (1) V is semistable (resp. stable) with respect to χ .
- (2) $\theta(W) \geq 0$ (resp. $\theta(W) > 0$) for every nonzero, proper, framed submodule W of V . Here W is called framed if W_0 is V_0 or zero.

Proof. The argument in [Ki] which uses Proposition 2.2 works with group G , too. We provide its detail. For (1) \Rightarrow (2), let W be a framed submodule of V . If $W_0 = V_0$, then there is a 1-parameter subgroup $\lambda : \mathbb{C}^\times \rightarrow G$ such that $V^{-1} = V$, $V^0 = W$ and $V^1 = 0$, where $V^n := \{p \in V : \lambda(t) \cdot p = t^n p, m \geq n, \forall t \in \mathbb{C}^\times\}$ (by the standard $G = \prod_{i \in Q'_0} GL_{v_i}(\mathbb{C})$ action on $V = \oplus_{i \in Q_0} \mathbb{C}^{\dim V_i}$). If $W_0 = \{0\}$, then there is a 1-parameter subgroup $\lambda : \mathbb{C}^\times \rightarrow G$ such that $V^0 = V$, $V^1 = W$ and $V^2 = \{0\}$. In either case, $\lim_{t \rightarrow 0} \lambda(t) \cdot V$ (by the action on the space of homomorphisms) exists and $\langle \chi, \lambda \rangle = \sum_{n \in \mathbb{Z}} n \theta(V^n/V^{n+1}) = \theta(W)$. Therefore, we conclude the proof by Proposition 2.2. For (2) \Rightarrow (1), suppose that $\lim_{t \rightarrow 0} \lambda(t) \cdot V$ exists for a nontrivial 1-parameter subgroup λ . This implies that V^n is a framed submodule of V for every n . Since the λ -action is nontrivial, some V^n is a proper, nonzero submodule of V . Now the proof follows from the identity $\langle \chi, \lambda \rangle = \sum_{n \in \mathbb{Z}} n \theta(V^n/V^{n+1}) = \sum_{n \in \mathbb{Z}} \theta(V^n)$. \square

This motivates the following. First, *from now on we assume that Q_0'' has only one vertex 0*. Let K be a vector space of rank r .

Definition 5.2. Denote by $\text{Rep}_\lambda(\overline{Q}, K)$ the category whose objects are finite dimensional $\mathbb{C}\overline{Q}/(\mu - \lambda)$ -modules V with an identification $V_0 = K^S$ for some finite (possibly empty) set S and whose morphisms, say from V to V' are module homomorphisms $(\varphi_i)_{i \in Q_0}$ satisfying the condition: $\varphi_0 : V_0 = K^S \rightarrow V'_0 = K^{S'}$ is a ‘block’ matrix *over* K , i.e., by definition, if $\varphi_0 = (\varphi_0^{s,s'})_{s \in S, s' \in S'}$, $\varphi_0^{s,s'}$ is a *multiplication* map for every pair (s, s') .

Proposition 5.3. $\text{Rep}_\lambda(\overline{Q}, K)$ is an abelian category.

Proof. The category is a subcategory of the abelian category of finite dimensional $\mathbb{C}\overline{Q}/(\mu - \lambda)$ -modules. Hence, it is enough to show that the (co)kernel map and the image map of a block matrix φ_0 is a block matrix. For the kernel (resp. the image) it can be seen by row (resp. column) operations for the block matrix $A := \varphi_0$. The cokernel map can be expressed by a block matrix C solving $CA' = 0$, where A' is the reduced matrix of A by column operations. \square

Define a homomorphism

$$Z : \mathbb{Z}^2 \rightarrow \mathbb{C}, \quad (x, y) \mapsto y + \sqrt{-1}x.$$

For $V \in \text{Rep}_\lambda(\overline{Q}, K)$, let $Z(V) = Z(v_0, v_1)$ where $v_0 = \dim V_0$, $v_1 = \dim V_1$ and $V_1 = \bigoplus_{i \in Q'_0} V_i$. Note that for nonzero V , $Z(V) \neq 0$ and $0 \leq \text{Arg}(Z(V)) \leq \pi/2$.

For a nonzero $V \in \text{Rep}_\lambda(\overline{Q}, K)$, if we take

$$\theta_0 := -v_1/v_0, \quad \theta_i := 1, \forall i \in Q'_0$$

then $\text{Arg}Z(W) \leq \text{Arg}Z(V)$ if and only if $\theta(W) \geq 0$ (resp. $\text{Arg}Z(W) < \text{Arg}Z(V)$ if and only if $\theta(W) > 0$.) for every nonzero proper subobject W of V in $\text{Rep}_\lambda(\overline{Q}, K)$. Therefore when $V_0 = K$, Bridgeland’s stability ([Br]) defined by the stability function Z coincides with King’s θ -stability (Theorem 5.1). From now on we will consider the GIT quotient variety $X // G$ with respect to linearization $(\theta_i)_{i \in Q'_0} = (1, \dots, 1)$.

5.2. Quiver sheaves. In this section, for every $i \in Q'_0$ fix $\lambda_i \in \Gamma(C, \omega_C)$. For a systematic study of stabilities on quasimaps we interpret quasimaps as linear objects.

Definition 5.4. A data $(E_i, \phi_a)_{i \in Q_1, a \in \overline{Q}_1}$ is called **M-twisted quiver sheaf** on C with respect to (\overline{Q}, λ) if E_i is a coherent sheaf on C ; E_0 is $K^S \otimes \mathcal{O}_C$ for some finite set S ; and ϕ_a is a \mathcal{O}_C -module homomorphism from E_{ta}

to $E_{ha} \otimes M_a$. The homomorphisms are subject to relation (which will be denoted also by $\mu - \lambda = 0$):

$$(5.1) \quad \sum_{i \in Q'_0} \sum_{a \in \overline{Q}_1: t\overline{a}=i} (-1)^{|a|} (\phi_a \otimes \text{Id}_{M_{\overline{a}}}) \circ \phi_{\overline{a}} - \text{Id}_{E_i} \otimes \lambda_i = 0,$$

unless stated otherwise.

Remark 5.5. For the history of the studies of (the moduli spaces of) twisted quiver sheaves usually without the relation, see [Al, AG, GK, Sh] and references therein. See also [Sz1].

Like a quiver representation and a \mathcal{O}_C -sheaf, a quiver sheaf can be considered as a module of the \mathbf{M} -twisted path algebra $\mathbf{M}\overline{Q}$ over \mathcal{O}_C ([GK, AG]). For each path $p = a_m \dots a_1$, let $M_p = M_{a_m}^\vee \otimes_{\mathcal{O}_C} M_{a_{m-1}}^\vee \otimes_{\mathcal{O}_C} \dots \otimes_{\mathcal{O}_C} M_{a_1}^\vee$ and for e_i , let $M_{e_i} = \mathcal{O}_C$. Let

$$\mathbf{M}\overline{Q}/(\mu - \lambda) = \left(\bigoplus_{\text{all paths } p} M_p \right) / (\mu - \lambda)$$

which has a \mathcal{O}_C -algebra structure similar to the path algebra $\mathbb{C}\overline{Q}$. Here $(\mu - \lambda)$ is the two-sided ideal generated by the relations (5.1) for ‘abstract’ $\phi_a \in M_a$: For every local section $\xi \in \omega_C^\vee$, consider

$$(\mu - \lambda)_i(\xi) := \sum_{ha=i} (-1)^{|a|} \xi_a \otimes \xi_{\overline{a}} - \langle \xi, \lambda_i \rangle e_i$$

where $\xi_a \otimes \xi_{\overline{a}}$ is an element in $M_a^\vee \otimes M_{\overline{a}}^\vee$ corresponding to ξ by the given isomorphism $M_a^\vee \otimes M_{\overline{a}}^\vee \cong \omega_C^\vee$. The ideal sheaf $\mu - \lambda$ is defined to be the ideal sheaf generated by $(\mu - \lambda)_i(\xi)$ for all $i \in Q'_0$, $\xi \in \omega_C^\vee$.

Given a $\mathbf{M}\overline{Q}/(\mu - \lambda)$ -module structure on E , a \mathbf{M} -twisted quiver sheaf can be associated by letting $E_i = M_{e_i}E$ and $(\phi_a)|_U(m_a \otimes s) = m_a s$ for an open set $U \subset C$, $m_a \in M_a^\vee(U)$, $s \in E_{ta}(U)$. (Here we regard $\phi_a : M_a^\vee \otimes E_{ta} \rightarrow E_{ha}$.) Conversely, a quiver sheaf defines a module structure on $\oplus E_i$.

There is a notion of isomorphisms between them. Note that, for $\lambda = 0$, upto isomorphisms, a \mathbf{M} -twisted quasimap to some $X//G$ with degree β amounts to a \mathbf{M} -twisted quiver bundle (i.e., a quiver sheaf with E_i being locally free sheaf for every i) with

$$\text{rank } E_i = v_i, \quad \deg E_i = \beta(\det_i),$$

where \det_i is the character of G given by the determinant of i -th general linear group. Denote by $\text{Rep}_C(\overline{Q}, \mathbf{M}, K)$ the abelian category of \mathbf{M} -twisted quiver sheaves E with framing $E_0 = K^S \otimes \mathcal{O}_C$ for some finite set S (depending on E). A morphism from (E_i, ϕ_a) to (E'_i, ϕ'_a) is, by definition, a collection $(\varphi_i)_{i \in Q_0}$ of \mathcal{O}_C -homomorphism $\varphi_i : E_i \rightarrow E'_i$

making $\phi'_a \circ \varphi_{ta} = (\varphi_{ha} \otimes 1_{M_a}) \circ \phi_a$ for every $a \in \overline{Q}_1$, satisfying the framing condition: $\varphi_0 : K^S \otimes \mathcal{O}_C \rightarrow K^{S'} \otimes \mathcal{O}_C$ is a block matrix of constant multiplications.

For $\delta > 0$, define a homomorphism $Z_\delta : \mathbb{Z}^3 \rightarrow \mathbb{C}$ by assignments

$$\begin{aligned} Z(v_0, v_1, d) &= Z_1(v_1, d) + Z_2(v_0, v_1), \\ Z_1(v_1, d) &= \frac{v_1}{2} + \sqrt{-1}d_1, \\ Z_2(v_0, v_1) &= \frac{v_1}{2} + \sqrt{-1}\delta v_0. \end{aligned}$$

Also define $Z_\delta(E) \in \mathbb{C}$ by the rank-degree map $\text{Rep}_{\lambda, C}(\overline{Q}, \mathbf{M}, K) \rightarrow \mathbb{Z}^3$, $E \mapsto (\text{rank} E_0, \text{rank} E_1, \deg E_1)$ followed by Z , where $E_0 = \oplus_{i \in Q'_0} E_i$, $E_1 = \oplus_{i \in Q''_0} E_i$.

The homomorphism (followed by $\pi/2$ -rotation) is a stability function on \mathbb{Z}^3 with Harder-Narasimhan property in the sense of Bridgeland by Proposition 2.4 (Artinian and Noetherian conditions together imply the HN property) in [Br]. Let $\mu_\delta(E) \in (-\infty, \infty]$ be the slope of $Z_\delta(E)$ for a nonzero quiver sheaf E . We abuse notation by letting μ stand for slopes as well as moment maps. This shouldn't cause any confusion.

Definition 5.6. A nonzero \mathbf{M} -twisted quiver sheaf E with $\text{rank} E_i \neq 0$ for some $i \in Q'_0$ is called δ -semistable (resp. δ -stable) if $\mu_\delta(E') \leq \mu_\delta(E)$ (resp. $\mu_\delta(E') < \mu_\delta(E)$) for every nonzero proper subobject E' of E in $\text{Rep}_{\lambda, C}(\overline{Q}, \mathbf{M}, K)$.

It is necessary that a semistable quiver sheaf is locally free. Let μ_i , $i = 1, 2$, be the slope function given by Z_i when $\delta = 1$.

Definition 5.7. Denote by $\langle E_0 \rangle$ the smallest quiver saturated subsheaf of E containing E_0 (which can be obtained by the intersection of all submodules F satisfying $E_0 \subset F$ and E/F is torsion free). The sheaf $\langle E_0 \rangle_i$ at vertex i is generated by global sections upto saturation. Hence, $\deg \Lambda^{\dim(E_0)_i} \langle E_0 \rangle_i \geq 0$. Here the saturation means that the sheaf $E_i / \langle E \rangle_i$ is torsion free for every i .

The following Lemma is on some facts which will be used in the proof of Proposition 5.11 below.

Lemma 5.8. *Fix the curve C and a covering map $\phi : C \rightarrow \mathbb{P}^1$. Let E be a vector bundle on C .*

- (1) *If $H^1(C, E \otimes \phi^* \mathcal{O}(m_0)) = 0$, then there is a number n_0 depending only on $\deg E$, $\text{rank} E$, and m_0 satisfying $H^1(C, E^\vee \otimes \phi^* \mathcal{O}(m)) = 0$ for all $m \geq n_0$.*
- (2) *If $H^1(C, E^\vee \otimes L) = 0$ for a line bundle L on C , then $\deg F \leq (|\deg L| + |1 - g|)\text{rank} E$ whenever F is a subsheaf of E .*

Proof. Let b be the degree of the covering map.

For (1): Since $0 = H^1(C, E \otimes \phi^* \mathcal{O}(m_0)) = H^1(\mathbb{P}^1, \phi_* E(m_0))$, $\phi_* E = \oplus \mathcal{O}(a_i^E)$ with $a_i^E + m_0 \geq -1$. Hence $\sum_i a_i^E \geq a_j^E - (m_0 + 1)br$ for any j . On the other hand, $\deg \phi_* E + br = \deg E + \text{rank} E(1 - g)$. Therefore, $a_j^E \leq \text{rank} E(1 - g) + m_0 br + \deg E$. Take $n_0 = |\text{rank} E(1 - g) + m_0 br + \deg E|$.

For (2): Note that $H^0(C, F \otimes \omega_C \otimes L^\vee) \subset H^0(C, E \otimes \omega_C \otimes L^\vee) = H^1(C, E^\vee \otimes L)^\vee = 0$, which implies that $0 \geq \chi(C, F \otimes \omega_C \otimes L^\vee) = \deg F + \text{rank} F(2g - 2 - \deg L) + \text{rank} F(1 - g)$. Hence, we conclude the proof. \square

Theorem 5.9. *Assume $\lambda = 0$. The moduli space*

$$Qmap_\delta(X // G, \mathbf{M}, \beta, C)$$

of δ -stable \mathbf{M} -twisted quiver bundles of degree β on C is a DM-stack of finite type over \mathbb{C} , equipped with a natural symmetric obstruction theory $(R^\bullet \pi_ \mathcal{E} \otimes \mathbf{M})^\vee$.*

Proof. Note that by Remark 3.7 and the fact that there are no nontrivial automorphisms of stable objects (except overall multiplications), the complex $(R^\bullet \pi_* \mathcal{E} \otimes \mathbf{M})^\vee$ is of amplitude contained in $[-1, 0]$. The complex can be expressed by a two-term perfect complex by the method in the proof of Corollary 4.4. Hence, as in Proposition 4.3, it is a symmetric obstruction theory for $Qmap_\delta(X // G, \mathbf{M}, \beta, C)$.

The construction of moduli stack $Qmap_\delta$ can be done by the argument parallel to the case $Qmap$ (i.e., the case when δ is large enough) which is handled in [CK, CKM], once we have the boundedness of $Qmap_\delta$. We will not repeat the procedure. We prove the boundedness using Harder-Narasimhan filtration with respect to the standard slope μ_{st} . Let (E, ϕ^a) be μ_δ -semistable quiver sheaf. Considering E as a sheaf $\oplus E_i$ on C , take the Harder-Narasimhan filtration

$$0 = E^0 \subset E^1 \subset \dots \subset E^l = E$$

of E for μ_{st} in the category of \mathcal{O}_C sheaves. Since HN filtration of a direct sum is a certain sum of each HN filtration, $E^i = \oplus_{j \in Q_0} E_j^i$, where E_j^i denotes $E^i \cap E_j$. Also, note that $E_0 = E_0^i / E_0^{i-1}$ for some i .

Claim: $\mu_{st}(E^i / E^{i-1}) \leq N_i(\mu_\delta(E), l, \deg \mathbf{M}) := \max\{0, \mu_\delta(E)\} + (l - i) \max\{0, \deg M_a \mid a \in \overline{Q}_1\}$.

We prove Claim by induction on $l - i$. When $i = l$, the claim is true since $\mu_{st}(E^l / E^{l-1}) \leq \mu_{st}(E^l)$. We define a composite $\psi_i^a : E^i \rightarrow E_{ta}^i \rightarrow E_{ha} \rightarrow E_{ha} \otimes M_a / E_{ha}^i \otimes M_a \rightarrow E \otimes M_a / E^i \otimes M_a$ where the first map is the natural monomorphism and the second map is the restriction of ϕ^a to E_{ta}^i , the third map is the projection, and the last

map is the natural monomorphism. Define $\psi_i = \bigoplus_{a \in \overline{Q}_1} \psi_i^a : \bigoplus_a E^i \rightarrow \bigoplus_a (E \otimes M_a / E^i \otimes M_a)$, where $\bigoplus_a E^i$ is the sum of \overline{Q}_1 -many copies of E^i . If $\psi_i = 0$ then E^i is a subobject of E in $\text{Rep}_{\lambda, C}(\overline{Q}, \mathbf{M}, K)$ since $E^j / E^{j-1} = E_0$ for some j . Hence, $\mu_1(E^i) := \frac{\deg E^i}{\text{rank } E^i} \leq \mu_\delta(E^i) \leq \mu_\delta(E)$ which, combined with $\mu_{st}(E^i / E^{i-1}) \leq \mu_{st}(E^i) = \mu_1(E^i) - \frac{\text{rank } E_0^i \deg E^i}{\text{rank } E^i \text{rank } E_1^i}$, implies $\mu_{st}(E^i / E^{i-1}) \leq 0$ if $\deg E^i \leq 0$ and $\mu_{st}(E^i / E^{i-1}) \leq \mu_\delta(E)$. Thus, $\deg E^i \leq \max\{0, \mu_\delta(E)\}$. If $\phi_i \neq 0$, then $\mu_{st}(E^{i'} / E^{i'-1}) \leq \mu_{st}(E^{i''} / E^{i''-1} \otimes M_a)$ for some a and $i' \leq i \leq i'' - 1$. By induction hypothesis, $\mu_{st}(E^{i'} / E^{i'-1}) \leq N_i(\mu_\delta(E), l, \deg \mathbf{M})$.

Let F be a nonzero subsheaf of E . If $0 = F^0 \subset F^1 \subset \dots \subset F^m = F$ is the HN filtration of F for μ_{st} , for every $1 \leq k \leq m$, the natural map $F^k / F^{k-1} \rightarrow E^i / E^{i-1}$ is nonzero for some i . Hence, $\mu_{st}(F^k / F^{k-1}) \leq \mu_{st}(E^i / E^{i-1})$ for some i , which implies, combined with the claim, that $\mu_{st}(F) \leq N_1(\mu_\delta(E), \text{rank}(E), \deg \mathbf{M})$. Now the boundedness follows from Theorem 1.1 in [Si]. \square

Remark 5.10. Note that the above proof of the boundedness holds also for δ -semistable \mathbf{M} -twisted quiver bundles of degree β on C .

Proposition 5.11. *Assume $\lambda = 0$. Fix $v = (v_i) \in \mathbb{N}_{>0}^{Q_0}$ and $d = (d_i) \in \mathbb{Z}^{Q_0}$. There is a number $\delta_0 > 0$ such that for all $\delta \geq \delta_0$, the following conditions are equivalent for \mathbf{M} -twisted quiver bundles E with numerical data (v, d) in the category $\text{Rep}_{\lambda, C}(\overline{Q}, \mathbf{M}, K)$ where $\dim_{\mathbb{C}} K = v_0$.*

- (1) δ -semistability.
- (2) δ -stability.
- (3) the stability as a \mathbf{M} -twisted quasimap to $X //_{\theta} G$ where $\theta = (1, \dots, 1)$.
- (4) $\langle E_0 \rangle = E$.

Proof. By the boundedness in previous Theorem, there is a number m_0 for which the condition $H^1(C, E \otimes \phi^* \mathcal{O}(m_0)) = 0$ in Lemma 5.8 holds, where $E = \bigoplus E_i$. Let $N = (|\deg L| + |1 - g|)\text{rank } E$ in Lemma 5.8 with $L = \phi^* \mathcal{O}(n_0)$. Take any number δ_0 such that

$$(5.2) \quad \delta_0 \text{Min}_{0 \neq v' < v} |\mu_2(v) - \mu_2(v')| > N + |\mu_1(E)|,$$

for any pair $v' = (v'_0, v'_1)$ of integers satisfying $v'_0 = v_0$ or 0 , $0 \leq v'_1 \leq v_1$ and $v' \neq v$.

(1) \Rightarrow (4): If $\langle E_0 \rangle \neq E$, then $\mu_2(\langle E_0 \rangle) > \mu_2(E)$, which implies that $\delta(\mu_2(E) - \mu_2(\langle E_0 \rangle)) < -|\mu_1(E)|$ by (5.2). Hence $\mu_\delta(E) < \mu_\delta(\langle E_0 \rangle)$ since $0 \leq \deg \langle E_0 \rangle_i$ for all i .

(4) \Leftrightarrow (3): This is because $\langle E_0 \rangle = E$ if and only if $\mu_2(\langle E_0 \rangle) \leq \mu_2(E)$ and $\langle E_0 \rangle|_p = \langle (E_0)|_p \rangle$ for general $p \in C$, where the latter is defined by the submodule of the path algebra $\mathbb{C}\overline{Q}/\mu - \lambda$ (after choosing identifications) generated by $(E_0)|_p$.

(3) \Rightarrow (2): Let E' be a nonzero quiver subsheaf of E where E is stable as a quasimap. Since $\mu_2(E') \leq \mu_2(E)$, which implies that $\mu_\delta(E') < \mu_\delta(E)$ by (5.2) and Lemma 5.8. \square

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SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, 87
 HOEGIRO, DONGDAEMUN-GU, SEOUL, 130-722, KOREA
E-mail address: bumsig@kias.re.kr